Limit

Limits

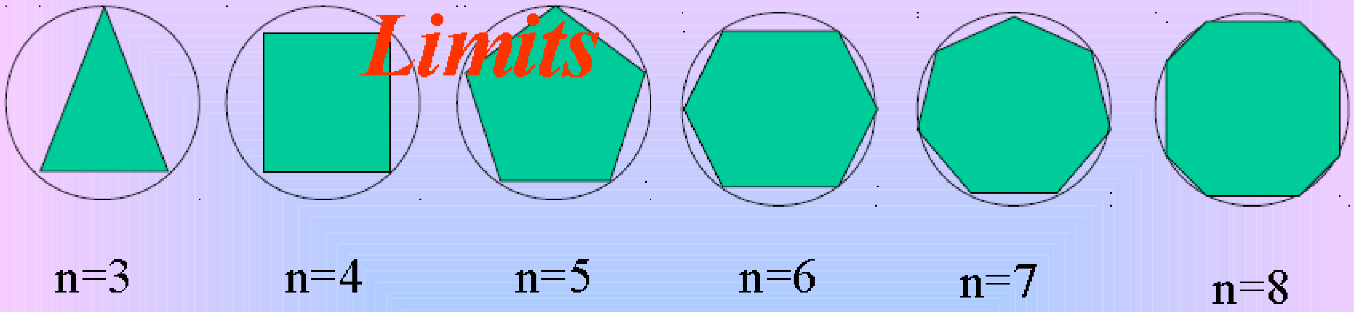
↓

“nearness”

The Idea of Limits

Consider a polygon inscribed in a circle

The Idea of Limits



'As number of sides of polygon increases, its area approximates the area of the circle'

'limit of Area of polygon is the Area of the circle'

As n approaches infinity ,

Lim Area of polygon = Area of the circle

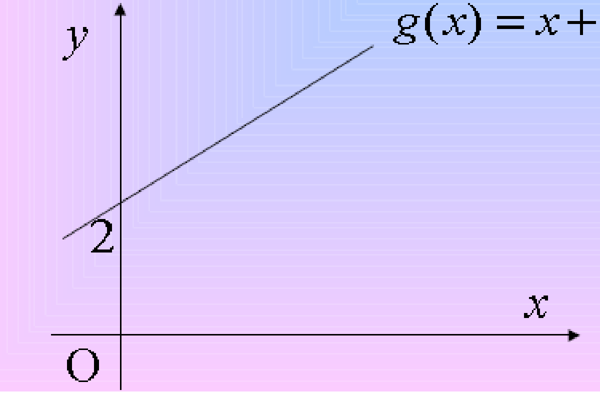
$$\lim_{x \to c} f(x) = L$$

The Idea of Limits

Consider the function:

$$g(x) = x + 2$$

x	1.9	1.99	1.999	1.9999	**2**	2.0001	2.001	2.01	2.1
$g(x)$	3.9	3.99	3.999	3.9999	4	3.0001	4.001	4.01	4.1



$$\lim_{x \to 2} g(x) = 4$$

As x approaches to positive 2 at both directions

Limit

Let f be a function defined on $S \subseteq R$ except possibly at $\in S$. A number l is said to be limit of function f as x tends to a if for a number $\in > 0$ there exist a positive number $\delta > 0$ such that

$$|f(x) - l| < \in \quad whenever \; |x - x_0| < \delta$$

Symbolically as $\lim_{x \to a} f(x) = l$

It means that

As $x \to a \Rightarrow f(x) \to l$

Answer of any Limit

(i) $\lim_{x \to a} f(x) = Finite$ (In this case Limit exist)

(ii) $\lim_{x \to a} f(x) = \infty \; or - \infty$ (In this case Limit does not exist)

Indeterminate Forms

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $0 \times \infty$ (iv) $\infty - \infty$

(v) 1^{∞} (vi) ∞^{0} (vii) 0^{0}

Existence of Limit

R.H.L. (Right Hand Limit):

$\lim_{x \to a^{+}} f(x) = \lim_{h \to 0} f(a+h)$ where $h > 0$ and $0 < h < 1$.

L.H.L. (Right Hand Limit):

$\lim_{x \to a^{-}} f(x) = \lim_{h \to 0} f(a-h)$ where $h > 0$ and $0 < h < 1$.

Definition of Limit

The limit of the function *f(x)* as *x* tends to *a* if both right and left hand exists and are equal. Their common value is called limit of *f(x)*.

In symbols

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$$

Some Important Formula

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(iii) $log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(iv) $log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

(v) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Example: Find the limit $\lim_{x\to 0} |x|$

Solution: Here, $f(x) = |x|$

Right Hand Limit

$$\lim_{x\to 0^+} |x| = \lim_{h\to 0} |0 + h| = \lim_{h\to 0} |h|$$

$$= \lim_{h\to 0} h = 0$$

Left Hand Limit

$$\lim_{x\to 0^-} |x| = \lim_{h\to 0} |0 - h| = \lim_{h\to 0} |-h|$$

$$= \lim_{h\to 0} h = 0$$

$Hence, \quad R.H.L. = L.H.L.$

$$Therefore\ \lim_{x\to 0}|x| = 0.$$

Example: Find the limit $\lim_{x\to 0}\frac{|x-1|}{x-1}$

Solution: Please try yourself.

Greatest Integer function or Bracket function

Let $x \in R$ then bracket function of x is denoted and defined as

$$[x] = \text{Greatest integer less than or equal to } x.$$

Example: [2.9]= 2, [1.3]= 1, [-1.3]= -2, [0]= 0, [-2.1]= -3

Example: Find the limit $\lim_{x\to 0}[x]$

Solution: Here, $f(x) = [x]$

Right Hand Limit

$$\lim_{x\to 0^+}[x] = \lim_{h\to 0}[0+h] = \lim_{h\to 0}[h]$$

$$= \lim_{h\to 0} 0 = 0$$

Left Hand Limit

$$\lim_{x\to 0^-}[x] = \lim_{h\to 0}[0-h] = \lim_{h\to 0}[-h]$$

$$= \lim_{h\to 0} -1 = -1$$

$$Hence, \quad R.H.L. \neq L.H.L.$$

$$Therefore \ \lim_{x\to 0}[x] = Does\ not\ exist.$$

Example: Find the limit $\lim_{x\to 0}(x - [x])$

Solution: Please try yourself.

Example: Let *f* be a function defined on R as

$$f(x) = \begin{cases} 1 - 2x,, & when \;\; x < 0 \\ 0, & when\; x = 0 \\ 1 + 3x, & when\; x > 0 \end{cases}$$

Find the limit of f(x) at *x = 0.*

Solution: At the point *x = 0* then

Right Hand Limit

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} 1 + 3(0 + h)$$

$$= \lim_{h \to 0} 1 + 3h = 1$$

Left Hand Limit

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} 1 - 2(0 - h)$$

$$= \lim_{h \to 0} 1 - 2h = 1$$

$$Hence, \quad R.H.L. = L.H.L.$$

$$Therefore \; \lim_{x \to 0} f(x) = 1.$$

Example: Let *f* be a function defined on [0,1] as

$$f(x) = \begin{cases} \frac{1}{2} - x, & 0 < x < \frac{1}{2} \\ \frac{3}{2} - x, & \frac{1}{2} < x < 1 \end{cases}$$

Find the limit of f(x) at $x = 1/2$.

Solution: Please try yourself.

Example: Let *f* be a function defined on R as

$$f(x) = \begin{cases} 1, & x \leq 0 \\ -1, & x > 0 \end{cases}$$

Find the limit of f(x) at $x = 0$.

Solution: Please try yourself.

Definition of Limit ($\in -\delta$)

Let f be a function defined on $S \subseteq R$ except possibly at $\in S$. A number l is said to be limit of function f as x tends to a if for a number $\in > 0$ there exist a positive number $\delta > 0$ such that

$$|f(x) - l| < \in \quad \mathit{whenever} \;\; |x - x_0| < \delta$$

Example: Prove that $\lim_{x \to 0}(3x - 1) = 5$ by $\in -\delta$ definition.

Solution: Let $\in > 0$ be given.

Then we have to find $\delta > 0$ (depending on $\in$) such that

$$|f(x) - 5| < \in \quad \mathit{whenever} \;\; 0 < |x - 2| < \delta$$

Now consider, $\qquad |f(x) - 5| < \in$

$$|3x - 1 - 5| < \in$$

$$|3x - 6| < \in$$

$$3|x-2| < \in$$

$$|x-2| < \in/3$$

Take $\delta = \in/3$

Then we get $0 < |x-2| < \delta$ then $|f(x)-5| < \in$

Hence, $\lim_{x\to 0}(3x-1) = 5$

Example: Let f be function defined by

$$f(x) = \frac{x^2-a^2}{x-a}; x \neq a$$

Using $\in -\delta$ definition show that $\lim_{x\to a}\frac{x^2-a^2}{x-a} = 2a$

Solution: Let $\in > 0$ be given.

Then we have to find $\delta > 0$ (depending on $\in$) such that

$$|f(x)-2a| < \in \quad whenever \;\; 0 < |x-a| < \delta$$

Consider, $|f(x) - 2a| < \in$

$$\left|\frac{x^2 - a^2}{x - a} - 2a\right| < \in$$

$$|x + a - 2a| < \in$$

$$|x - a| < \in$$

Take $\delta = \in$

Then we get $0 < |x - a| < \delta$ then $|f(x) - 2a| < \in$

Hence, $\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a$

Example: Prove that $\lim_{x \to 0} x \sin\frac{1}{x} = 0$ by $\in - \delta$ definition.

Solution: Please try yourself.

Theorem: If $\lim_{x\to a} f(x) = l$ exists then it is unique.

Proof: Let if Possible, f have two distinct limits $l_1 \ and\ l_2 \ at\ a$.

We choose $\in = \frac{1}{2}|l_1 - l_2| such\ that\ \in > 0$

Since, $\lim_{x\to a} f(x) = l_1$ then

for any $\in > 0\ \exists \delta_1\ such\ that\ 0 < |x-a| < \delta_1$

$$then\ |f(x) - l_1| < \in \quad \ldots.(1)$$

Since, $\lim_{x\to a} f(x) = l_2$ then

for any $\in > 0\ \exists \delta_2\ such\ that\ 0 < |x-a| < \delta_2$

$$then\ |f(x) - l_2| < \in \quad \ldots.(2)$$

Now, we put $\delta = \max\{\delta_1, \delta_2\}\ then\ 0 < |x-a| < \delta$

Now, $|l_1 - l_2| = |(f(x) - l_2) - (f(x) - l_1)|$

$$\leq |f(x) - l_2| + |f(x) - l_1|$$

$$< \in + \in$$

$$< 2 \in$$

$$|l_1 - l_2| < |l_1 - l_2|$$

Which is absurd, so our assumption is wrong.

Hence, $\lim_{x\to a} f(x)$ is unique.

Algebra of Limit

Let f and g are two a function defined on $S \subseteq R$ of a such that

$$\lim_{x\to a} f(x) = l,\ \lim_{x\to a} g(x) = m$$

(i) $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m$

(ii) $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l - m$

(iii) $\lim_{x\to a}(f(x) \,.\, g(x)) = \lim_{x\to a} f(x) \,.\, \lim_{x\to a} g(x) = l.m$

(iv) $\lim_{x\to a}(f(x) \,/ g(x)) = \lim_{x\to a} f(x) \,/ \lim_{x\to a} g(x) = l/m$

Provided $g(x) \neq 0$ and $m \neq 0$.

Theorem: If $\lim_{x\to a} f(x) = l$ then $\lim_{x\to a} |f(x)| = |l|$ but converse is not true.

Proof: Since, $\lim_{x\to a} f(x) = l$ then for any $\in > 0$ there exist a positive number $\delta > 0$ such that

$$|f(x) - l| < \in \quad whenever \ |x - a| < \delta$$

Using this inequality $$|a - b| \geq ||a| - |b||$$

Now, $$|f(x) - l| \geq ||f(x)| - |l||$$

So, $$||f(x)| - |l|| \leq \in \quad whenever \ |x - a| < \delta$$

Hence, $\lim_{x\to a} |f(x)| = |l|$

For Example

$$f(x) = \begin{cases} 1, & x \leq 0 \\ -1, & x > 0 \end{cases} \text{ at } x = 0$$

$\lim_{x\to 0} f(x)$ *does not exists* but $|f(x)| = 1$

So, $\lim_{x\to 0} |f(x)| = 1$ exists.

Continuity

Continuity

A function *f(x)* is said to be continuous at a point $x = a$ if

- (i) *f(a)* is defined.
- (ii) $\lim_{x \to a} f(x)$ exist
- (iii) $\lim_{x \to a} f(x) = f(a)$

Remark: If any one of the above condition does not satisfy then we say *f(x)* is discontinuous at $x = a$.

Example: (i) Discuss the continuity of $f(x) = \frac{1}{x} \quad at \;\; x = 0$

Solution: Here, $f(a) = f(0) = \frac{1}{0} = \infty = not\ defined$

Hence, *f(x)* is not continuous at $x = 0$.

Example: (ii) Discuss the continuity of $f(x) = x^2 + 5x \ \ at \ \ x = 2$

Solution: Here, (i) $f(a) = f(2) = 4 + 10 = 14$

(ii) $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 + 5x = 14$

(iii) $\lim_{x\to 2} f(x) = f(a) = 14$

Hence, *f(x)* is continuous at *x* = *2.*

Example: (iii) Discuss the continuity of $f: R \to R$ such that

$$f(x) = \begin{cases} \frac{\sin 2x}{x}; x \neq 0 \\ 1; \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Solution: Here, (i) $f(a) = f(0) = 1$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{x} = 2$

(iii) $\lim_{x\to 0} f(x) \neq f(a)$

Hence, *f(x)* is not continuous at *x* = *0.*

Continuity (By L.H.L. and R.H.L.)

A function $f(x)$ is said to be continuous at a point $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

i.e. $$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

Example: Let f be a function defined on R as

$$f(x) = \begin{cases} 2x + 3,, & when \ \ x > 0 \\ 0, & when\ x = 0 \\ x + 3, & when\ x < 0 \end{cases}$$

Discuss the continuity of $f(x)$ at $x = 0$.

Solution: At the point $x = 0$ then

Right Hand Limit

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} 2(0 + h) + 3$$

$$= \lim_{h\to 0} 2h + 3 = 3$$

Left Hand Limit

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}(0-h)+3$$

$$= \lim_{h\to 0} -h + 3 = 3$$

$Here, \quad R.H.L. = L.H.L.$

$Now, \qquad f(a) = f(0) = 0$

So, $R.H.L. = L.H.L. \neq f(a)$

Hence, *f(x)* is not continuous at *x = 0*.

Example: Let *f* be a function defined on R as

$$f(x) = \begin{cases} \frac{\log(1+4x)-\log(1+3x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Find the value of *k* if *f(x)* is continuous at *x* = 0.

Solution: Please try yourself.

Definition of Continuity ($\in -\delta$)

A function *f(x)* is said to be continuous at a point $x = a$ if for any number $\in > 0$ there exist a positive number $\delta > 0$ such that

$$|f(x) - f(a)| < \in \quad whenever \ |x - x_0| < \delta$$

Example: Prove that $f(x) = (3x - 1)$ is continuous at *x* = 2 by $\in -\delta$ definition.

Solution: Let $\in > 0$ be given.

Then we have to find $\delta > 0$ (depending on $\in$) such that

$$|f(x) - 5| < \in \quad whenever \;\; 0 < |x - 2| < \delta$$

Now consider, $|f(x) - 5| < \in$

$$|3x - 1 - 5| < \in$$

$$|3x - 6| < \in$$

$$3|x - 2| < \in$$

$$|x - 2| < \in/3$$

Take $\delta = \in/3$

Then we get $0 < |x - 2| < \delta$ then $|f(x) - 5| < \in$

Hence, *f(x)* is continuous at $x = 2$.

Algebra of Continuous Function

Let f and g are two a function defined on $S \subseteq R$ of a such that

$f(x)$ and $g(x)$ are continuous at $x = a$.

(i) $f(x) + g(x)$ are continuous at $x = a$.
(ii) $f(x) - g(x)$ are continuous at $x = a$.
(iii) $f(x) \,.\, g(x)$ are continuous at $x = a$.
(iv) $f(x) / g(x)$ are continuous at $x = a$.
Provided $g(x) \neq 0$

Theorem: If $f(x)$ is continuous at $x = a$. then $|f(x)|$ is also continuous at $x = a$ but converse is not true.

Proof: Since, $f(x)$ is continuous at $x = a$ then for any $\in > 0$ there exist a positive number $\delta > 0$ such that

$$|f(x) - f(a)| < \in \quad whenever \ |x - a| < \delta$$

Using this inequality $|a-b| \geq ||a|-|b||$

Now, $|f(x)-f(a)| \geq ||f(x)|-|f(a)||$

So, $||f(x)|-|f(a)|| \leq \in \quad whenever \ |x-a|<\delta$

Hence, $|f(x)|$ is also continuous at $x = a$

For Example

$$f(x)=\begin{cases}1, & x \leq 0\\ -1, & x>0\end{cases} \text{ at } x = 0$$

$\lim_{x \to 0} f(x) \ does\ not\ exists$ but $|f(x)|=1$

So, $\lim_{x \to 0} |f(x)|=1$ exists.

Example: Let *f* be a function defined on R as

$$f(x)=\begin{cases} -x, & when\ \ x<-1 \\ 1, & when-1\le x\le 1 \\ x, & when\ x>1 \end{cases}$$

Discuss the continuity of *f(x)* at *x = 1 and -1*.

Solution: Please try yourself.

Example: Let *f* be a function defined on R as

$$f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}, & x\neq 0 \\ 1,\ \ x=0 \end{cases}$$

Discuss the continuity of *f(x)* at *x =0*.

Solution: Please try yourself.

Continuity in Open Interval

A function *f* is said to be continuous in open interval if it is continuous at every point (a, b).

Example: (i) $f(x) = x^2$ is continuous at (0, 2).

(ii) $f(x) = \frac{1}{x}$ is not continuous at (-1, 1).

Continuity in Closed Interval

A function *f* is said to be continuous in closed interval [a, b]

if it is

(i) Continuous at every point (a, b).
(ii) Continuous from the right at a.
(iii) Continuous from the left at b.

Example: Consider $f(x) = x^2 + 2x \ \ in \ [1,2]$

(i) $f(x) = x^2 + 2x$ is continuous in (1, 2).

(ii) Right Hand Limit at $x = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h)^2 + 2(1+h)$$
$$= 3$$

(iii) Left Hand Limit at $x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} (2-h)^2 + 2(2-h)$$
$$= 8$$

Hence, $f(x)$ is continuous at [1, 2].

Types of Discontinuity

There are three types of discontinuity

- (i) Removable Discontinuity
- (ii) Discontinuity of First kind (Jump Discontinuity)
- (iii) Discontinuity of Second kind

Removable Discontinuity

A function $f(x)$ is said to be removable discontinuity at $x = a$. If

- (i) $f(a)$ is defined.
- (ii) $\lim_{x \to a} f(x)$ exist
- (iii) $\lim_{x \to a} f(x) \neq f(a)$

Example: Which type of discontinuity of $f: R \to R$ such that

$$f(x) = \begin{cases} \frac{\sin x}{x}; x \neq 0 \\ 0; \quad x = 0 \end{cases} \text{ at } x = 0.$$

Solution: Here, (i) $f(a) = f(0) = 0$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x\to 0} f(x) \neq f(a)$

Here, *f(x)* has removable discontinuity at $x = 0$.

Discontinuity of First Kind

A function *f(x)* is said to be discontinuity of first kind at $x = a$. If

R.H.L. = L.H.L. exist but R.H.L. ≠ L.H.L.

Example: Which type of discontinuity of the function $f(x) = [x]$ at $x = 0$.

Solution: Here, $f(x) = [x]$

Right Hand Limit

$$\lim_{x \to 0^+} [x] = \lim_{h \to 0} [0 + h] = \lim_{h \to 0} [h]$$

$$= \lim_{h \to 0} 0 = 0$$

Left Hand Limit

$$\lim_{x \to 0^-} [x] = \lim_{h \to 0} [0 - h] = \lim_{h \to 0} [-h]$$

$$= \lim_{h \to 0} -1 = -1$$

$Here,$ R.H.L. and L.H.L. exists but

$R.H.L. \neq L.H.L.$

Here, *f(x)* has discontinuity of first kind at $x = 0$.

Discontinuity of Second Kind

A function *f(x)* is said to be discontinuity of first kind at $x = a$. If

R.H.L. and L.H.L. does not exist.

Example: Which type of discontinuity of $f: R \to R$ such that

$$f(x) = \begin{cases} \sin\frac{1}{x}; x \neq 0 \\ 0; \quad x = 0 \end{cases}$$ at $x = 0$.

Solution: Here, (i) $f(a) = f(0) = 0$

(ii) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin\frac{1}{x} = does\ noe\ exists$

Here, $f(x)$ has discontinuity of second kind at $x = 0$.

Differentiability

Differentiability

A function *f(x)* is said to be differentiable at a point $x = a$ if

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

OR

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

And the limit is called the derivative of f at a point a and denoted by $f'(a)$.

Example: Discuss the differentiability of $f(x) = x^2 \quad at \; x = 0$

Solution: Given, $f(x) = x^2 \; at \; x = 0$

Now,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 - 0}{x} = \lim_{x\to 0} x = 0$$

Hence, *f(x)* is differentiable at $x = 0$.

A necessary condition for the existence of a finite derivative

Statement: If function is differential at a point then it is continuous at that point but not conversely.

Proof: Given, *f(x)* is differentiable at $x = a$.

Aim: *f(x)* is continuous at $x = a$.

Given *f(x)* is differentiable at $x = a$ then by definition of differentiable

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{exist and finite} \quad \ldots(1)$$

Now,

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a}(x - a)$$

Taking both sides limit at $x = a$.

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} (x - a) \right]$$

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right] . \lim_{x \to a} (x - a)$$

$$\lim_{x \to a} [f(x) - f(a)] = f'(a). 0$$

$$\lim_{x \to a} [f(x) - f(a)] = 0$$

Then

$$\lim_{x \to a} f(x) = f(a)$$

Hence, $f(x)$ is continuous at $x = a$.

The converse may or may not be true.

For example, Let $f(x) = |x| \; at \; x = 0$ is continuous.

Now,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{|x| - 0}{x} = \lim_{x\to 0} \frac{|x|}{x}$$

Right Hand Limit

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{h\to 0} \frac{|0 + h|}{0 + h} = \lim_{h\to 0} \frac{h}{h}$$

$$= \lim_{h\to 0} 1 = 1$$

Left Hand Limit

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{h\to 0} \frac{|0 - h|}{0 - h} = \lim_{h\to 0} \frac{h}{-h}$$

$$= \lim_{h\to 0} -1 = -1$$

So, $R.H.L. \neq L.H.L.$

Hence, *f(x)* is not differentiable at *x* = 0.

Example: Discuss the differentiability of

$$f(x) = \begin{cases} x\sin\frac{1}{x}\ ; x \neq 0 \\ 0; \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Solution: Given, $f(x) = \begin{cases} x\sin\frac{1}{x}\ ; x \neq 0 \\ 0; \quad x = 0 \end{cases} at\ \ x = 0$

Now,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x\sin\frac{1}{x} - 0}{x} = \lim_{x \to 0} \sin\frac{1}{x} = Does\ not\ exist$$

Hence, *f(x)* is not differentiable at $x = 0$.

Example: Discuss the differentiability of

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} \; ; x \neq 0 \\ \quad 0; \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Solution: Please try yourself.

Existence of Derivative

R.H.D. (Right Hand Derivative)[$Rf'(a)$]:

$$Rf'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

where $h > 0$ and $0 < h < 1$.

L.H.D. (Right Hand Derivative) [$Lf'(a)$]:

$$Lf'(a) = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

where $h > 0$ and $0 < h < 1$.

Differentiability (By L.H.D. and R.H.D.)

A function *f(x)* is said to be differentiable at a point *x* = *a* if

(i) $Rf'(a)\ \ and\ \ Lf'(a)$ both are exist.

(ii) $Rf'(a)\ =\ Lf'(a)$

Example: Discuss the differentiability of

$$f(x) = \begin{cases} x^2 - 1, & x \geq 1 \\ 1 - x, & x < 1 \end{cases} \text{ at } x = 1$$

Solution: Here, $f(a) = f(1) = 0$

Now, R.H.D

$$Rf'(a) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1 - 0}{h}$$

$$= \lim_{h\to 0} \frac{1 + h^2 + 2h - 1}{h} = \lim_{h\to 0}(h + 2) = 2$$

L.H.D

$$Lf'(a) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{1 - (1-h) - 0}{-h}$$

$$= \lim_{h\to 0} \frac{1 - 1 + h}{-h} = \lim_{h\to 0} -1 = -1$$

Therefore, $R.H.D. \neq L.H.D.$

Hence, *f(x)* is not differentiable at *x* = 1.

Example: Discuss the differentiability of

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 2 - 3x + x^2, & x \geq 2 \end{cases} \text{ at } x = 2$$

Solution: Please try yourself

Example: Discuss the differentiability of

$$f(x) = \sin|x| - |x| \text{ at } x = 0$$

Solution: Here, $f(a) = f(0) = 0$

Now, R.H.D

$$Rf'(a) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sin|0+h| - |0+h| - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin h - h}{h} = \lim_{h\to 0} \left[\frac{\sin h}{h} - 1\right] = 1 - 1 = 0$$

L.H.D

$$Lf'(a) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{\sin|0-h| - |0-h| - 0}{-h}$$

$$= \lim_{h\to 0} \frac{\sin h - h}{-h} = \lim_{h\to 0} \left[\frac{\sin h}{-h} + 1\right] = -1 + 1 = 0$$

Therefore, $R.H.D. = L.H.D.$

Hence, *f(x)* is differentiable at *x* = 0.

Example: Let *f* be a function defined on R as

$$f(x) = \begin{cases} \dfrac{x}{1+e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Discuss the differentiability of *f(x)* at *x =0.*

Solution: Please try yourself.